401. Multiplying out and splitting up the fraction:

$$\begin{aligned} y &= \frac{2x^3+8x^2+10x+4}{\sqrt{x}} \\ &\equiv 2x^{\frac{5}{2}} + 8x^{\frac{3}{2}} + 10x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}. \end{aligned}$$

Differentiating,

$$\frac{dy}{dx} = 5x^{\frac{3}{2}} + 12x^{\frac{1}{2}} + 5x^{-\frac{1}{2}} - 2x^{-\frac{3}{2}}.$$

402. By the factor theorem, $y = k(x-3)(x+4)$ for some constant k. Substituting $x = 0, y = 6$ gives $k = -\frac{1}{2}$. Multiplying out, $y = -\frac{1}{2}x^2 - \frac{1}{2}x + 6$.

403. A fraction in its lowest terms is zero if and only if its numerator is zero. The indices aren't relevant. The roots are $x = a$ and $x = -b$.

404. Neither of the quantities x^4 or x^2 can be negative, so the implication goes both ways: ① $\iff$ ②.

405. This is a quadratic in $\sqrt{p}$, with coefficients a, b, c. So, according to the quadratic formula,

$$\sqrt{p} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

Squaring both sides,

$$p = \left(\frac{-b \pm \sqrt{b^2-4ac}}{2a}\right)^2.$$

406. Replacing x by $x-4$ gives $(x-6)^2 + (y-3)^2 = 1$.

407. (a) Trivially, this is 0.2×10^n. But this isn't in standard form, so we move a factor of ten from the right to the left. This gives $2 \times 10^{n-1}$.
(b) We get 0.75×10^{-5}, which, in standard form, is 7.5×10^{-6}.

408. Factorising the equations given, we see that only

$$\begin{aligned} y &= x^4 - x^2 \\ &\equiv x^2(x+1)(x-1) \end{aligned}$$

has a double (and not triple) root at zero. Hence, only (b) could be the equation of the graph.

409. The statement is phrased in terms of implication, so "and" is wrong. The word should be "or". On the other hand, rephrasing the whole statement with reference to roots would mean that "and" is fine. Either of the following is logically correct:

① "The equation $x(x-1) = 0$ implies that $x = 0$ or $x = 1$."

② "The roots of the equation $x(x-1) = 0$ are $x = 0$ and $x = 1$"

410. Using *suvat*, we have an object dropped from rest, so, neglecting air resistance, $s = \frac{1}{2}gt^2$. In this case, displacement until the splash is depth d, and $g \approx 10$, so $d \approx 5t^2$, as required.

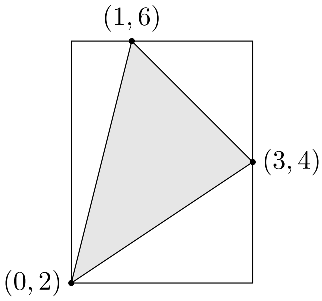
411. Forming an equation for surface area,

$$\begin{aligned} & 2(x \cdot 2x + x \cdot 3x + 2x \cdot 3x) = 88 \\ \implies & x^2 = 4. \end{aligned}$$

Since the length x must be positive, the solution is $x = 2$.

412. Statement (a) is false. There are infinitely many x values which give $\sin x = \frac{1}{2}$. A counterexample to the rightwards implication is $x = 150°$.

413. We enclose the triangle in a rectangle:



The rectangle has area 12, from which we must subtract triangles with areas $2, 2, 3$. This leaves an area of 5.

414. Removing the three-way intersection, the strictly two-way intersections have probabilities

$$\begin{aligned} \mathbb{P}(X \cap Y \cap Z') &= \tfrac{2}{3} - \tfrac{1}{2} = \tfrac{1}{6}, \\ \mathbb{P}(Y \cap Z \cap X') &= \tfrac{2}{3} - \tfrac{1}{2} = \tfrac{1}{6}, \\ \mathbb{P}(Z \cap X \cap Y') &= \tfrac{2}{3} - \tfrac{1}{2} = \tfrac{1}{6}. \end{aligned}$$

The probability of at least two events occurring is the sum of the above plus $\mathbb{P}(X \cap Y \cap Z) = \frac{1}{2}$. This gives a total of 1. Hence, it is guaranteed that at least two of the events happen.

415. Completing the square,

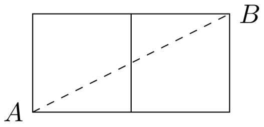
$$\begin{aligned} y &= ax^2 + bx + c \\ &\equiv a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c. \end{aligned}$$

The presence of the squared bracket means that the algebra is symmetrical around

$$x = -\frac{b}{2a}.$$

So, x values either side of $x = -\frac{b}{2a}$ must produce the same y value. Hence, the parabola has a line of symmetry at $x = -\frac{b}{2a}$. QED.

416. Unfolding the two faces to a flat rectangle, the length of the path is given by the hypotenuse of a right-angled triangle with sides of length 1 and 2.



Hence, the path has length $\sqrt{5}$.

417. The fractions inside the the brackets are the same, in fact, because $(x-a) \equiv -(a-x)$. Hence, we can subtract the powers by the usual index law. This gives a power of -1, which is a reciprocal. The full simplification is

$$\begin{aligned} &\left(\frac{x-a}{x-b}\right)^{-\frac{1}{3}} \div \left(\frac{a-x}{b-x}\right)^{\frac{2}{3}} \\ \equiv &\left(\frac{x-a}{x-b}\right)^{-\frac{1}{3}} \div \left(\frac{x-a}{x-b}\right)^{\frac{2}{3}} \\ \equiv &\left(\frac{x-a}{x-b}\right)^{-1} \\ \equiv &\frac{x-b}{x-a}. \end{aligned}$$

418. Multiplying out and comparing coefficients, we get $p^2 = 4$, $-2pq = -12$, and $q^2 = r$. Since $p, q, r > 0$, we must have $p = 2, q = 3$ and $r = 9$.

419. (a) The function is $\text{f}(x) = 4x + 7$.
(b) f is linear, so its derivative is constant. Hence, there is no need to consider a limiting process, as the gradient for any $\text{h} \neq 0$ is the same as the gradient of the line $y = 4x + 7$.
(c) Using a gradient triangle between points $(x, 4x+7)$ and $(x+h, 4(x+h)+7)$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{4(x+h)+7-(4x+7)}{h} \\ &\equiv \frac{4x+4h+7-4x-7}{h} \\ &\equiv \frac{4h}{h} \\ &\equiv 4, \text{ since } h \neq 0. \end{aligned}$$

So, the gradient of $y = 4x + 7$ is 4.

420. (a) $60°$ per second is $\frac{1}{6}$ of a revolution per second, which is 10 rpm.
(b) 240π radians per hour is 120 revolutions per hour, which is 2 rpm.

421. The output transformation is a stretch scale factor k in the y direction.

The input transformation is a replacement of the input $(x-a)$ by $(x-b)$, which is equivalent to a replacement of the variable x by $\big(x-(b-a)\big)$. This is a translation by vector $(b-a)\mathbf{i}$.

422. (a) $\text{f}\left(\frac{7}{3}\right) = 0$.
(b) By the factor theorem, $(3x-7)$ is a factor. Taking this out,

$$\begin{aligned} &6x^3 - 23x^2 + 24x - 7 \\ \equiv &(3x-7)(2x^2-3x+1) \\ \equiv &(3x-7)(2x-1)(x-1). \end{aligned}$$

— Alternative Method —

The polynomial long division calculation to take the factor out is

$$\begin{array}{r} 2x^2 \quad - 3x + 1 \\ 3x-7 \overline{) \; 6x^3 - 23x^2 + 24x - 7} \\ \underline{-6x^3 + 14x^2 \qquad\qquad\quad} \\ -9x^2 + 24x \qquad \\ \underline{9x^2 - 21x \qquad} \\ 3x - 7 \\ \underline{-3x + 7} \\ 0 \end{array}$$

423. Since the result is at least three, the possibility space is restricted to $\{3, 4, 5, 6\}$. These outcomes are all equally likely. So, $p = \frac{1}{2}$.

424. (a) Since the programmer is only concerned with quadrilaterals, the domain should be the set of all quadrilaterals.
(b) Any set containing the numbers $0, 1, 2$ will serve as a codomain. The obvious candidate is the set of non-negative integers $\mathbb{Z}^+$, but e.g. $\{0, 1, 2\}$, $\mathbb{Z}$ or $\mathbb{R}$ are also correct.
(c) The range is precisely the outputs attainable with the given domain. A quadrilateral can have a maximum of 2 pairs of parallel sides, so the range is $\{0, 1, 2\}$.

425. 0! is defined to be 1. Also, from the unit circle, $(\cos 0, \sin 0)$ is the point $(1, 0)$. So, we have

$$\left.\frac{x! + \sin x}{x! + \cos x}\right|_{x=0} = \frac{1+0}{1+1} = \frac{1}{2}.$$

— Nota Bene —

The definition of 0! to be 1 is for ease of use in formulae such as

$${}^n\text{C}_r \equiv \frac{n!}{r!(n-r)!}.$$

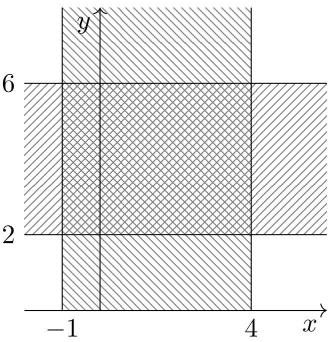
Considering Pascal's triangle, ${}^n\text{C}_0$ must be equal to 1, which is only true if we define $0! = 1$.

426. The inequality $X^2 < X$ has boundary equation $X^2 - X = 0$, which has roots $X = 0$ and $X = 1$. The inequality has solution $X \in (0, 1)$. This is an interval of length 1. The possibility space for X is $[-2, 2]$, which is an interval of length 4. So, $\mathbb{P}(X^2 < X) = \frac{1}{4}$.

Nota Bene

The inclusion or exclusion of the endpoints of the interval $(0, 1)$ is not relevant to this calculation: the interval $[-2, 2]$ is continuous, so the probability of any particular value is zero. So, e.g. $\mathbb{P}(X^2 \leq X)$ would also be $\frac{1}{4}$.

427. The region is a rectangle, consisting of all points below which are shaded twice:



The width is 5 and height 4, so the area is 20.

428. (a) This is true. The factor a is common to all terms and can be taken out.
(b) This is false. The term b is added to all terms, so, if taken out, it should be $n \times b$.

429. Using the binomial expansion, the coefficient is

$$^5\text{C}_4(3)^4(-1)^1 = -405.$$

Nota Bene

The answer here should be -405 as opposed to $-405x^4$. If the question had asked for the *term* in x^4, as opposed to the *coefficient* of x^4, the correct answer would have been $-405x^4$.

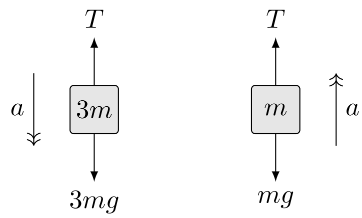
430. The worded statements translate algebraically as $x = p\sqrt[3]{t}$ and $y^2 = mt + b$, where p, m and b are constants. Rearranging the first equation to make t the subject, we get $t = qx^3$, where q is a new constant. Substituting for t gives $y^2 = mqx^3 + b$. Renaming mq as a, $y^2 = ax^3 + b$, as required.

431. Factorising,

$$\begin{aligned} & 8 \cdot (2^x)^2 - 33 \cdot (2^x) + 4 = 0 \\ \Longrightarrow\ & (8 \cdot 2^x - 1)(2^x - 4) = 0 \\ \Longrightarrow\ & 2^x = 1/8, 4. \end{aligned}$$

So, the solution is $x = -3, 2$.

432. (a) i. For the tension applied by the string to be constant, the string must be light and the pulley smooth,
ii. For the magnitudes of the accelerations to be equal, the string must be inextensible.
(b) Force diagrams:



(c) Vertical $F = ma$ gives $3mg - T = 3ma$ and $T - mg = ma$. Adding these gives $2mg = 4ma$, so $a = \frac{1}{2}g$. Then $T = \frac{3}{2}mg$.
(d) Clearly acceleration would be reduced. Also, the tension would no longer be the same on both sides of the pulley: the tension acting on the larger mass would be larger and the tension acting on the smaller mass would be smaller.

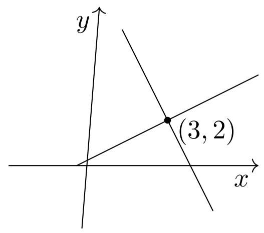
433. The index law $\left(a^b\right)^c \equiv a^{bc}$ gives

$$\left(a^3\right)^x \equiv a^{3x} \equiv \left(a^x\right)^3 = z^3.$$

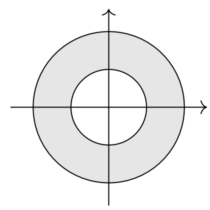
434. Assume, for a contradiction, that quadrilateral Q has four acute interior angles. An acute angle is $\theta < 90°$, so the four angles $\theta_1 + \theta_2 + \theta_3 + \theta_4 < 360°$. This is a contradiction, since the interior angles of a quadrilateral add to 360°.

Hence, every quadrilateral must have an interior angle that is not acute. QED.

435. These are perpendicular lines, intersecting at the point $(3, 2)$:



436. The annular region $1 \geq x^2 + y^2 \geq 4$ is a ring. It consists of a disc of radius 2 centred at the origin, minus a disc of radius 1 centred at the origin.

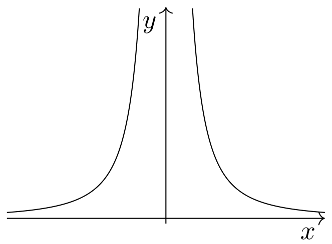


The area is $4\pi - 1\pi = 3\pi$.

437. The coordinate axes are perpendicular to the unit circle, so this is true.

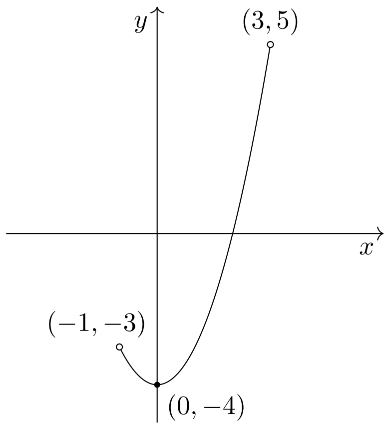
438. (a) $\Delta = 8^2 - 4 \cdot 2p \cdot (2p + 15)$, which simplifies to $64 - 120p - 16p^2$.
    (b) We are told that the original quadratic has exactly one root, so set $\Delta = 0$. Factorising, $(8 + p)(8 - 16p) = 0$, so $p = -8, 1/2$.

439. This is $y = \frac{1}{x^2}$, which is broadly akin to $y = \frac{1}{x}$, except that negative values of $\frac{1}{x}$ are then squared to give positive y values. So, the graph is



440. (a) $[0, 2) \cap [1, 3) = [1, 2)$,
    (b) $(0, 2) \cap [1, 3] = [1, 2)$,
    (c) $[0, 2] \cap (1, 3) = (1, 2]$.

441. The domain includes the vertex of the parabola, which is the minimum $(0, -4)$.



The maximum is given, then, by the higher of the values at the extremes of the domain, which must, by symmetry, be $f(3) = 5$. The range is therefore $[-4, 5)$.

442. (a) $F = 3 + 4 = 7$ N, so $a = \frac{7}{5}$ ms^{-2}.
    (b) By Pythagoras, $F = 5$ N, so $a = 1$ ms^{-2}.
    (c) $F = 4 - 3 = 1$ N, so $a = \frac{1}{5}$ ms^{-2}.

443. Such an equilateral triangular prism has three square and two equilateral triangular faces. The squares have total area 3; the equilateral triangles have total area $\sqrt{3}/2$. So the total surface area is $A = 3 + \sqrt{3}/2$.

444. Subtracting the two equations, $(x + 3)^2 - x^2 = 9$. This gives $6x + 9 = 9$, so $x = 0$. Substituting this in, $y = \pm 1 - 2$. The coordinates, therefore, are $(0, -1)$ and $(0, -3)$.

— Nota Bene —

It is always worth taking a moment to see whether an equation or set of equations has any common features or symmetries. In this case, both contain $(y + 2)^2$. This offers a shortcut as above. If you missed this, multiplying everything out brute-force style would be a long way round.

445. By symmetry, the cross-section $XAYC$ is a square. XY is the diagonal of a square of length l, so $|XY| = \sqrt{2}l$.

446. (a) By the factor theorem, $f(x)$ must have factors of $(x - 3)$ and $(x + 2)$. This gives the equation as $y = k(x - 3)(x + 2)$. Multiplying out the linear factors gives the required result.
    (b) Substituting $(2, -16)$ gives $k = 4$.

447. It makes no difference what the first roll is. The probability that the second is different is then $\frac{5}{6}$.

448. (a) Displacement is the integral of velocity. So, the integrand $2t + 5$ is the velocity. Initial velocity is 5, the acceleration is 2, and the duration is $3 - 0 = 3$.
    (b) Integrating gives a displacement of 24. (There are no units in the question, so this quantity is unitless.)

449. Replacing x by kx is a stretch factor $1/k$ in the x direction. There is no change in the y direction. So, the area scale factor between R_1 and R_2 is $1/k$.

450. Since the quadratic $x^2 + x - p$ has a factor of $x - q$, we know that $x = q$ is a root. So, $q^2 + q - p = 0$. This gives $p = q^2 + q$.

451. The vectors $\mathbf{i}$ and $\mathbf{k}$ are unit vectors in the x and z directions, and are therefore perpendicular. So, by Pythagoras, $v = \sqrt{1 + 3} = 2$ ms^{-1}.

452. Since $y = x + k$ is a line, k must be a constant. Thus $x = k + 1$ and $y = k + 1$ are horizontal and vertical lines. So $y = x + k$, which has gradient 1, must intersect them both.

453. Adding all three equations eliminates T_1 and T_2 directly. This gives $5mg = 20ma$. So, $a = \frac{1}{4}g$.

454. Since $g(0) = 2$, we know that $g(x) = ax + 2$:

$$\begin{aligned} &\int_0^3 ax + 2\,dx = 6 \\ \Longrightarrow\ &\left[\tfrac{1}{2}ax^2 + 2x\right]_0^3 = 6 \\ \Longrightarrow\ &\tfrac{9}{2}a + 6 = 6 \\ \Longrightarrow\ &a = 0. \end{aligned}$$

Hence, g is constant: $g(x) = 2$.

—— Nota Bene ——

The term "linear" gets used to refer to polynomials of degree 1 and degree 0, both of which produce straight line graphs.

Degree	Polynomial
0	Constant/linear
1	Linear
2	Quadratic
3	Cubic
4	Quartic

455. (a) The periods are

i. $360°$ and $180°$,

ii. $60°$ and $45°$.

(b) The period of such a sum is given by the lcm of the individual periods; $\text{lcm}(60, 45) = 180$, so G has period $180°$.

456. If the minimum value of $g(x)$ occurs at $x = \alpha$, then $g(x)$ must be stationary there, so $g'(\alpha) = 0$. This means $f(\alpha) = 0$, which in turn tells us that f has a root at α. □

457. Differentiating,

$$\begin{aligned} &y = x^3 \\ \Longrightarrow\ &\frac{dy}{dx} = 3x^2. \end{aligned}$$

At $x = 6/5$, the gradient is $108/25$. The gradient of the normal, therefore, is $-25/108$. The equation of the normal is then

$$y - 1.2^3 = -\tfrac{25}{108}(x - 1.2).$$

Substituting $x = 0$, we get $y \approx 2.0057$. So, the normal crosses the y axis close to $y = 2$.

458. 3D Pythagoras gives $d = \sqrt{2^2 + 3^2 + 6^2} = 7$.

459. Integrating gives $f(x) = mx + c$. So, the family of solutions is the set of straight lines of gradient m.

460. By the factor theorem, $(x-2)$ is a factor. We take this out in the usual way. The first term of the remaining quadratic must be $8x^2$:

$$8x^3 - 22x^2 + 13x - 2 \equiv (x-2)(8x^2 + ...)$$

The by-product is $-16x^2$. We want $-22x^2$, so the next term must produce $-6x^2$. So, it is $-6x$:

$$8x^3 - 22x^2 + 13x - 2 \equiv (x-2)(8x^2 - 6x + ...).$$

The by-product is $12x$. We want $13x$, so the next term must produce x. So, it is 1. This gives the correct constant term -2, as expected:

$$8x^3 - 22x^2 + 13x - 2 \equiv (x-2)(8x^2 - 6x + 1).$$

Factorising the quadratic,

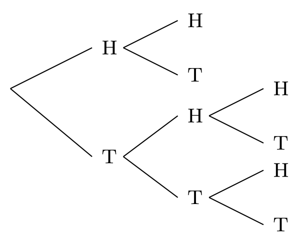
$$8x^3 - 22x^2 + 13x - 2 \equiv (x-2)(4x-1)(2x-1).$$

—— Nota Bene ——

The polynomial long division to take out the factor $(x-2)$ is:

$$\begin{array}{r|l} & 8x^2 \quad - 6x + 1 \\ \hline x - 2) & 8x^3 - 22x^2 + 13x - 2 \\ & -8x^3 + 16x^2 \\ \hline & -6x^2 + 13x \\ & 6x^2 - 12x \\ \hline & x - 2 \\ & -x + 2 \\ \hline & 0 \end{array}$$

461. (a) The tree diagram, all of whose branches have probability $1/2$, is as follows:



(b) The successful outcomes are HH and THH. So, the probability is $(1/2)^2 + (1/2)^3 = 3/8$.

462. Reflecting in the line $y = x$ switches the x and y coordinates. This gives $x = ay^2 + by + c$.

463. NII horizontally is $Q - \frac{1}{3}P = 2$ and NII vertically is $\frac{1}{4}P - (Q - 6) = 0$. Adding the equations gives $P = 48$, then $Q = 18$.

464. A regular n-gon can be split up into $n-2$ triangles. Each contributes π radians. Hence, the sum of the interior angles is $(n-2)\pi$. Dividing by n gives

$$\theta = \frac{(n-2)\pi}{n}.$$

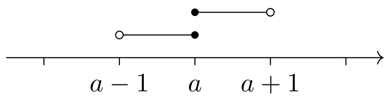
465. The board is the possibility space here. Counting outcomes, 36 squares of 64 are away from the edge. Hence, the probability is $\frac{36}{64} = \frac{9}{16}$.

466. The gradient is $\dfrac{\Delta y}{\Delta x} = \dfrac{b-a}{a-b} \equiv -\dfrac{a-b}{a-b} \equiv -1.$

467. Any two functions differing by a constant give a counterexample: say $f(x) = \sin x$, $g(x) = \sin x + 1$. These are not identical, but they have identical derivatives $\cos x$.

468. (a) $A_\triangle = \frac{1}{2}ab\sin C$ gives $\frac{1}{2} \cdot 13^2 \cdot \sin\theta = 60$. So, $\theta = \arcsin \frac{120}{169}$ or $\theta = 180° - \arcsin\frac{120}{169}$.
(b) The cosine rule gives $n^2 = 2 \cdot 13^2 - 2 \cdot 13 \cdot \cos\theta$. Plugging in the two possible angles and taking the positive square root, we get $n = 10, 24$.

469. The two intervals overlap at a, so the union is $(a-1, a+1)$:



470. (a) $u_y = 196\sqrt{3}\sin 60° = 294 \text{ ms}^{-1}$.
(b) At the highest point, the vertical velocity is zero. Assuming projectile motion, acceleration is g downwards. So, for the greatest height attained, we use $v^2 = u^2 + 2as$. This gives $0 = 294^2 - 2gh$. With $g = 9.8$, we get $h = 4410$ m, which is 4.41 km.

471. Assume, for a contradiction, that f is quadratic, so $f(x) = ax^2 + bx + c$. Differentiating twice gives $f''(x) = 2a$, which is constant. So, if $f''(k) = 0$ for some $k \in \mathbb{R}$, then a must be zero. But if $a = 0$, f is not a quadratic function, which is a contradiction. So, f cannot be quadratic. QED.

472. Even though the cards are picked out together, we can still label one of them as the first and the other as the second. For both (a) and (b), there is only one successful route through the tree diagram, so we simply multiply probabilities:

(a) $\frac{4}{52} \times \frac{3}{51} = 0.00452$ (3sf).
(b) $\frac{26}{52} \times \frac{25}{51} = 0.245$ (3sf).

473. The interval (a, b) is contained in the interval $[a, b]$. This gives the following results, with $x = a$ and $x = b$ as counterexamples to the false statements:

(a) False.
(b) True.
(c) True.
(d) False.

— Nota Bene —

In fact, the two true statements are equivalent to one another, as are the two false statements.

474. Substituting $x = -2/3$ and $y = 0$,

$$\begin{aligned} 0 &= \tfrac{4}{9}p + \tfrac{2}{3} - 2 \\ \Longrightarrow p &= 3. \end{aligned}$$

Then substituting $x = q$, we get $3q^2 - q - 2 = 0$. This factorises as $(3q+2)(q-1) = 0$. We are looking for the root that is not $-2/3$, so $q = 1$.

475. (a) The factor of 3 can be taken out:

$$\int_a^b 3\,f(x)\,dx = 3\int_a^b f(x)\,dx = 3k.$$

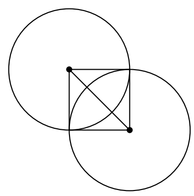
(b) We can integrate term by term:

$$\begin{aligned} &\int_a^b 2x - f(x)\,dx \\ \equiv &\int_a^b 2x\,dx - \int_a^b f(x)\,dx \\ \equiv &\Big[x^2\Big]_a^b - k \\ \equiv &\, b^2 - a^2 - k. \end{aligned}$$

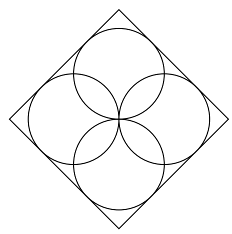
476. This is a quadratic in x^2. Completing the square gives $6(x^2+2)^2 - 24 + 13$, which simplifies to $6(x^2+2)^2 - 11$.

477. (a) $1 + 2 + 3 + 4 + 5 = 15$, and $\frac{1}{2} \cdot 5 \cdot 6 = 15$.
(b) Adding the equations gives $2S_n = n(n+1)$, since each pair of terms on the RHS adds to $n + 1$. Dividing by 2 gives the required result.

478. (a) Adjacent circles intersect each other at right angles. So, the distance between their centres is given by Pythagoras to be $\sqrt{2}$.



This distance is two radii minus the overlap, which has been counted twice. So, the overlap is $2 - \sqrt{2}$, as required.
(b) The minimal bounding square is as shown:



Each circle has diameter 2. From part (a), the length of the overlap is $2 - \sqrt{2}$. So, the edge length of the square is $4 - \big(2 - \sqrt{2}\big)$, which is $2 + \sqrt{2}$. Squaring this gives $6 + 4\sqrt{2}$.

479. The cosine rule is $c^2 = a^2 + b^2 - 2ac\cos C$. In a right-angled triangle, $C = 90°$, and $\cos 90° = 0$. Hence, the cosine rule reduces to $c^2 = a^2 + b^2$, which is Pythagoras's theorem.

480. Reflection is enacted by switching the variables x and y. Then we add 5 to y. Overall, this gives

$$x = 3 - 2\lambda, \quad y = 4\lambda + 5, \quad \text{for } \lambda \in [0, 4].$$

481. Multiplying out gives $4mn + 2m + 2n$, which we can factorise as $2(2mn + m + n)$. Since $m, n \in \mathbb{N}$, $(2mn+m+n)$ is an integer greater than 1. Hence, the number cannot be prime. □

482. Take out the factor of x first, leaving a quartic:

$$x(x^4 - 8x^2 + 16) = 0$$

The quartic is a quadratic in x^2:

$$x(x^2 - 4)(x^2 - 4) = 0.$$

Hence, the solution is $x = 0, \pm 2$.

483. Multiplying out the brackets,

$$\begin{aligned} &\int \tfrac{5}{16}\left(2\sqrt{t}\right)^3 dt \\ = &\int \tfrac{5}{2} t^{\frac{3}{2}}\, dt \\ = &\ t^{\frac{5}{2}} + c. \end{aligned}$$

484. Dividing top and bottom by x,

$$\lim_{x\to\infty} \frac{6x+1}{2x-11} = \lim_{x\to\infty} \frac{6 + 1/x}{2 - 11/x}.$$

As $x \to \infty$, the inlaid fractions both tend to zero. The limit is therefore $\frac{6}{2} = 3$, as required.

485. The equation of the hypotenuse is $y = 5 - \frac{1}{2}x$. The equation of OP, since it is perpendicular to this, is $y = 2x$. Solving simultaneously gives $2x = 5 - \frac{1}{2}x$, so P is $(2, 4)$.

— Alternative Method —

All three triangles in the diagram are similar. The scale factor between the perpendicular sides is 2. So, point P divides the hypotenuse in the ratio $1 : 4$. Taking a weighted average of the coordinates of its endpoints, P is at $\left(\frac{1}{5} \times 10, \frac{4}{5} \times 2\right)$, which is $(2, 4)$.

486. Testing values, there are roots at $x = \pm a$. So, by the factor theorem, there are factors of $(x \mp a)$. Factorising, the equation is $(x + a)^2(x - a) = 0$. The factor of $(x + a)$ is squared, so $x = -a$ is the double root.

487. The perimeter of the hexagon is $6l$, where l is edge length. An equilateral triangle of edge length l has area $\frac{1}{4}\sqrt{3}l^2$, so the hexagon has area

$$\begin{aligned} A &= \tfrac{3\sqrt{3}}{2} l^2 \\ &\equiv \tfrac{3\sqrt{3}}{2}\left(\tfrac{1}{6}l\right)^2. \end{aligned}$$

This gives $24\,A = \sqrt{3}\,P^2$. QED.

488. Negative numbers have real cube roots, but do not have real square or fourth roots.

(a) $[0, \infty)$,

(b) $\mathbb{R}$,

(c) $[0, \infty)$.

— Nota Bene —

The "largest" real domain could well be called "broadest". Technically, by saying that D is the largest possible real domain, we mean that there is no other real domain E which contains D as a proper subset.

489. Multiplying by $(x - 4)$,

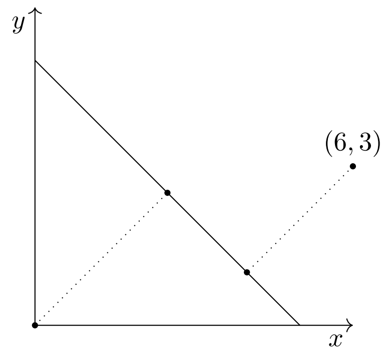
$$\begin{aligned} &(ax + b)(x - 4) + c \equiv 2x^2 - x - 16, \\ \implies &ax^2 + (b - 4a)x + (c - 4b) \equiv 2x^2 - x - 16. \end{aligned}$$

Equating coefficients of x^2, $a = 2$. Then, equating coefficients of x, $b = 7$. Then, equating constant terms, $c = 12$.

490. (a) This is correct. Both sides of the identity are positive and have the same magnitude.

(b) This is incorrect. For any positive x, the LHS is positive, while the RHS is negative.

491. The shortest distance to the line $y = 5 - x$ lies along a line perpendicular to $y = 5 - x$, i.e. a line of gradient 1.



The relevant points on $y = 5 - x$ are $(5/2, 5/2)$ and $(4, 1)$. The squared distances, along the dotted lines above, are $25/2$ and 8. So, $(6, 3)$ is closer.

492. (a) The distribution is symmetrical around the mean. So, $\mathbb{P}\big(X \in [-1,0]\big) \approx 0.34$.

(b) The set $\{0,1\}$ consists of the points 0 and 1. And individual points have probability zero in continuous distributions: $\mathbb{P}\big(X \in \{0,1\}\big) = 0$.

(c) $\mathbb{P}\big(X \in (0,1)\big) \approx 0.34$, for the same reason as in part (b).

(d) Doubling the area, $\mathbb{P}\big(X \in [-1,1]\big) \approx 0.68$.

493. Carrying out the definite integration,

$$\begin{aligned} &\int_0^k 6x^2 + 2\,dx = 20 \\ \Longrightarrow\ &\Big[2x^3 + 2x\Big]_0^k = 20 \\ \Longrightarrow\ &k^3 + k - 10 = 0. \end{aligned}$$

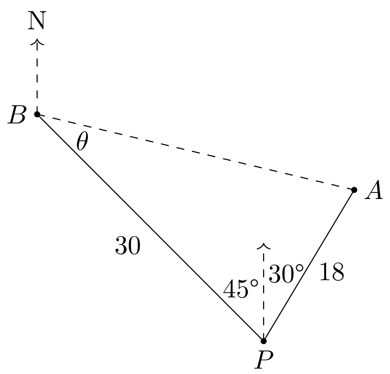
Spotting the integer root, or finding it with any numerical method, $k = 2$. Taking out the factor $(x-2)$, we are left with $k^2 + 2k + 5 = 0$. This has discriminant $\Delta = -16 < 0$. So, the quadratic has no real roots, and $k = 2$ is the only possibility.

— Alternative Method —

Polynomial long division gives

$$\begin{array}{r} x^2 + 2x \ \ + 5 \\ x - 2\,\big)\ \ x^3 + 0x^2 \ \ + x - 10 \\ -\,x^3 + 2x^2 \qquad\qquad \\ 2x^2 \ \ + x \qquad \\ -\,2x^2 + 4x \qquad \\ 5x - 10 \\ -\,5x + 10 \\ 0 \end{array}$$

494. After 90 minutes, the ships are as follows:

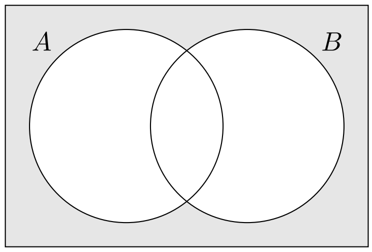


By the cosine rule, $|AB| = 30.732319....$ Then, by the sine rule,

$$\sin\theta = \frac{18\sin 75°}{30.732319...}.$$

This gives $\theta = 34.45°$. Hence, the bearing of A from B is given by $180° - 45° - \theta = 100.5°$ (1dp).

495. The statement is true. $A' \cap B'$ means everything that is simultaneously in not-A and not-B.



The above is the complement of the union, which is notated $(A \cup B)'$.

496. Substituting $x = w + 3$,

$$\begin{aligned} &x^2 + 12x \\ =\ &(w+3)^2 + 12(w+3) \\ \equiv\ &w^2 + 18w + 45. \end{aligned}$$

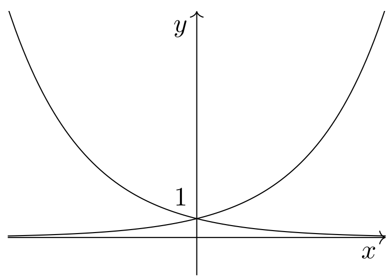
497. Since the acceleration is in the direction of $\mathbf{i} - 2\mathbf{j}$, and its component parallel to y has magnitude 14 ms^{-2}, the acceleration must be $7\mathbf{i} - 14\mathbf{j}$. So, the magnitude of the acceleration is

$$a = \sqrt{7^2 + 14^2} = 7\sqrt{5} \text{ ms}^{-2}.$$

Then NII gives

$$F = 0.01 \times 7\sqrt{5} = 0.157 \text{ N (3sf)}.$$

498. The first graph is a standard exponential. In the second graph, x has been replaced by $-x$, which reflects in the x direction, i.e. in the y axis.



499. (a) No. A counterexample is $x^2 - 1/2 = 0$, which has two real roots, and $2x^2 + x + 1/2 = 0$, which has none.

(b) Yes. Graphically, the new LHS is a translation of the old LHS by -1 in the x direction. So, this equation has the same set of roots as the original one, translated by -1.

(c) No. Solving this as a quadratic in x^2 must give $x^2 = a$ and $x^2 = b$, where a and b are the roots of the original equation. But if a and b are both negative or both positive, then this produces either zero or four real roots.

500. Substituting in $t = 0$ and $y = \sqrt{2}$, the endpoints of the line segment are at $(0, 5)$ and $\left(\sqrt{2}, 5 - \sqrt{2}\right)$. The distance between these, by Pythagoras, is

$$d = \sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2} = 2.$$

——— Alternative Method ———

For a unit change in the parameter t, position in each coordinate changes by 1. So, position along the line segment changes by $\sqrt{2}$. The parameter t takes values over an interval of length $\sqrt{2}$, so position changes by $\sqrt{2} \times \sqrt{2} = 2$.

——— End of 5th Hundred ———